NIDS Session 8

$\boxed{1}$ Consider the Hamiltonian system

$$\begin{cases} y'(t) = J^{-1} \nabla H(y(t)) = f(y) & (1) \\ y(0) = y_0 \end{cases}$$

and the variational problem

$$\psi'(t) = \frac{\partial f}{\partial y} \psi(t)$$

$$\psi(0) = I_{2n}$$

From (1) $\quad f(y) = J^{-1} \nabla H(y)$

$$\Rightarrow \frac{\partial f}{\partial y}(y) = J^{-1} \underbrace{\nabla^2 H(y)}_{\text{Hessian of } H}$$

We must show that $\operatorname{tr}\left(\frac{\partial f}{\partial y}\right) = \operatorname{tr}\left(J^{-1} \nabla^2 H(y)\right) = 0$

- First note that J is skew symmetric and so is J^{-1}

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad J^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

- Secondly, if H is C^2, the Hessian $\nabla^2 H$ is symmetric

General result : if $A \in M_n(K)$ is skew symmetric (for a field K) $B \in M_n(K)$ is symmetric

Then $\operatorname{tr}(AB) = 0 \qquad (2)$

Proof: First note that for $A, B \in M_n(\mathbb{R})$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

Indeed

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki} a_{ik} = \mathrm{tr}(BA)$$

To prove (2), note that if A is skew symmetric (i.e. $A^T = -A$) and B is symmetric (i.e. $B^T = B$)

$$\mathrm{tr}(AB) = \mathrm{tr}((AB)^T) = \mathrm{tr}(-BA) = -\mathrm{tr}(BA) = -\mathrm{tr}(AB)$$

$$\Rightarrow \mathrm{tr}(AB) = 0$$

Application:

$$\mathrm{tr}\left(\frac{\partial f}{\partial y}\right) = \mathrm{tr}(J^{-1} \nabla^2 H) = 0$$

By the Abel-Liouville-Jacobi formula applied to the variational problem

$$\frac{d}{dt}(\det(\Psi(t))) = \underbrace{\mathrm{tr}\left(\frac{\partial f}{\partial y}\right)}_{=0} \det(\Psi(t)) = 0$$

$\Rightarrow$ $\det(\Psi(t))$ is a first integral and

$$\det(\Psi(t)) = \det(\Psi(0)) = \det(I_{2m}) = 1$$

ii) $Vol(\varphi_t(\Omega)) = \int_{\varphi_t(\Omega)} dy = \int_\Omega \left| \det\left(\frac{\partial \varphi_t(y_0)}{\partial y_0} \right) \right| dx$

$$= \int_\Omega \underbrace{|\det(\Psi(t))|}_{=1} dx$$

$$= \int_\Omega dx = Vol(\Omega)$$

Thus φ_t is volume preserving

iii) Observation : $\operatorname{div}(b(y)) = \operatorname{tr}\left(\frac{\partial b}{\partial y} \right)$

$\boxed{\Leftarrow}$ From questions i) and ii), if $\operatorname{div}(b(y)) = 0$ then φ_t is volume preserving

$\boxed{\Rightarrow}$ Assume φ_t is volume preserving, meaning for every open bounded set Ω

$$Vol(\varphi_t(\Omega)) = \int_{\varphi_t(\Omega)} dy = \int_\Omega dx = Vol(\Omega)$$

$$= \int_\Omega \left| \det\left(\frac{\partial \varphi_t(y_0)}{\partial y_0} \right) \right| dx$$

Assume $\exists\, g_0^*$ such that $\left| \det\left(\frac{\partial \varphi_t(g_0^*)}{\partial g_0} \right) \right| \neq 1$

(Assume without loss of generality that

$$\left| \det\left(\frac{\partial \varphi_t}{\partial g_0}(g_0^*) \right) \right| > 1$$

If $\det\left(\frac{\partial \varphi_t}{\partial g_0}(g_0) \right)$ is continuous (admitted)

then $\exists U \subset \Omega$ such that $\left| \det\left(\frac{\partial \varphi_t}{\partial g_0}(g) \right) \right| > 1$

$$\forall g \in U$$

$$\Rightarrow \quad Vol(\varphi_t(U)) = \int_U \left| \det\left(\frac{\partial \varphi_t}{\partial g_0}(g_0) \right) \right| dx$$

$$> \int_U dx = Vol(U)$$

$\Rightarrow$ Contradiction

So $\left| \det\left(\frac{\partial \varphi_t}{\partial g_0}(g_0) \right) \right| = 1$ for all g_0, t

From the initial condition and the continuity of $\left|\det\left(\frac{\partial \varphi_t}{\partial y_0}(y_0)\right)\right|$, we deduce that

$$\det\left(\frac{\partial \varphi_t}{\partial y_0}(y_0)\right) = \det\left(\frac{\varphi_0}{\partial y_0}(y_0)\right) = \det(I_{2n}) = 1$$

From the variational equation

$$\frac{d}{dt}\Bigg(\underbrace{\det\left(\frac{\partial \varphi_t}{\partial y_0}(y_0)\right)}_{\psi(t)}\Bigg) = 0 = tr\left(\frac{\partial f}{\partial y}\right)\underbrace{\det(\psi(t))}_{=1}$$

$$\Rightarrow \quad tr\left(\frac{\partial f}{\partial y}\right) = 0$$

$\boxed{2}$ Consider a smooth transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$

$\boxed{\Rightarrow}$ Assume g is symplectic :

$$g'(y)^T J g'(y) = J$$

$$\Rightarrow \quad \det(g'(y))^2 = 1 \quad \Rightarrow \quad |\det(g'(y))| = 1$$

g is volume preserving

Since we are in $\mathbb{R}^2$, denoting $G = g'(y)$

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$G^T J G = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_{11} g_{22} - g_{12} g_{21} \\ -(g_{11} g_{22} - g_{12} g_{21}) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \det(G) \\ -\det(G) & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J}$$

because g is symplectic

$\Rightarrow$ $\det(G) = 1$ and $g(y)$ is both volume and orientation preserving

$\boxed{\Leftarrow}$ If $g(y)$ is volume and orientation preserving

$$\mathrm{Vol}(g(\Omega)) = \int_\Omega |\det(g'(y))| \, dy = \int_\Omega dy \qquad (1)$$

$$= \int_\Omega \det(g'(y)) \, dy$$

Using a similar argument as in $\boxed{2}$, iii)

(4) $\Rightarrow$ $\det(g'(y)) = 1$

From the first part

$\Rightarrow$

$$g'(y)^T J g'(y) = \begin{pmatrix} 0 & \det(g'(y)) \\ -\det(g'(y)) & 0 \end{pmatrix}$$

Because $n = 1$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

by definition

So, for $n = 1$, volume and orientation preservation implies symplecticity

ii) What happens when $n > 1$

$$G = g'(y) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

In this case,

$$G^T J G = \begin{pmatrix} G_{11}^T G_{21} - G_{21}^T G_{11} & G_{11}^T G_{22} - G_{21}^T G_{12} \\ G_{12}^T G_{21} - G_{22}^T G_{11} & G_{12}^T G_{22} - G_{22}^T G_{12} \end{pmatrix}$$

Counter-example:

Consider $G = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$\left(\text{i.e. } g(q_1, q_2) = \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} \right)$

$$\det(G) = (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

For n even, g is both volume and orientation preserving

yet $G^T J G = -J \neq J$

So $g(q)$ is not symplectic!

$\boxed{3}$ See the solution on the black board.